Journal of Approximation Theory **109**, 229–241 (2001)
doi:10.1006/jath.2000.3547, available online at http://www.idealibrary.com on IDEAL®

Upper Estimates in Direct Inequalities for Bernstein-Type Operators[1]

José A. Adell and C. Sangüesa[2]

Departamento de Métodos Estadísticos, Facultad de Ciencias, Universidad de Zaragoza, 50009 Zaragoza, Spain
E-mail: adell@posta.unizar.es, csangues@posta.unizar.es

Communicated by Dany Leviatan

Received October 11, 1999; accepted in revised form November 6, 2000; published online February 16, 2001

We obtain explicit upper estimates in direct inequalities with respect to the usual sup-norm distance for Bernstein-type operators. Our approach combines analytical and probabilistic techniques based on representations of the operators in terms of stochastic processes. We illustrate our results by considering some classical families of operators, such as Weierstrass, Szàsz, and Bernstein operators. © 2001 Academic Press

Key Words: direct inequality; Bernstein-type operator; Ditzian–Totik modulus of smoothness; upper estimate; stochastic process.

1. INTRODUCTION

It is well known that, for many families $\mathbb{L} = (L_\delta, 0 < \delta < \delta_0)$ of positive linear operators, the rate of convergence of $L_\delta f$ to f, as $\delta \to 0$, in the L_p-norm $(1 \leqslant p \leqslant \infty)$ is characterized in terms of a suitable weighted Ditzian–Totik modulus of smoothness of the function f under consideration (cf. [6, 7, 13] and the references therein). Results of this kind are referred to as direct and converse inequalities. In this paper, we combine analytical and probabilistic techniques to give general pointwise estimates concerning direct inequalities in the usual sup-norm for Bernstein-type operators.

We shall use the following notations. Let I be a closed real interval. Denote by $C(I)$ the set of real continuous functions defined on I. Following [6], a function $\varphi \in C(I)$ is called a weight function if $\varphi(x) > 0$, $x \in I^0$, where I^0 stands for the interior set of I. Given a weight function φ, we consider the following Ditzian–Totik modulus of smoothness of $f \in C(I)$,

$$\omega_\varphi^2(f; \varepsilon)_\infty := \sup\{|\Delta^2_{h\varphi(x)} f(x)| : 0 \leqslant h \leqslant \varepsilon, B(x, h\varphi(x)) \subseteq I\}, \qquad \varepsilon \geqslant 0,$$

[1] Research supported by Grant PB98-1577-C02-01 of the Spanish DGICYT.
[2] Supported by Grant CB 31/99 of the DGA-CAI-CONSI$_+$D.

0021-9045/01 $35.00

where $B(x, h)$ denotes the closed ball of center x and radius h and

$$\Delta_h^2 f(x) := f(x+h) - 2f(x) + f(x-h), \qquad h \geqslant 0, \quad B(x, h) \subseteq I.$$

Let φ be a weight function and let $\delta_0 > 0$. We consider a family $\mathbb{L} := (L_\delta, 0 < \delta < \delta_0)$ of positive linear operators of the form

$$L_\delta f(x) := Ef(Z_\delta(x)), \qquad x \in I, \quad 0 < \delta < \delta_0, \tag{1}$$

E being the mathematical expectation and $(Z_\delta(x),\ x \in I,\ 0 < \delta < \delta_0)$ a family of I-valued random variables satisfying for all $x \in I$ and $0 < \delta < \delta_0$

$$Z_\delta(x) = x + \delta\varphi(x)\, Y_\delta(x), \qquad EY_\delta(x) = 0, \qquad \text{and}$$
$$\sup_{0<\delta<\delta_0} \sup_{x \in I} EY_\delta^2(x) < \infty. \tag{2}$$

Many classical families of positive linear operators allow for this probabilistic expression (cf. [1]). Finally, we denote by $\mathscr{M}(\varphi)$ the set of functions $f \in C(I)$ satisfying $L_\delta\, |f|\, (x) < \infty$, $x \in I$, $0 < \delta < \delta_0$, as well as $\omega_\varphi^2(f; \varepsilon)_\infty < \infty$, $\varepsilon \geqslant 0$.

We shall be concerned with pointwise estimates of the type

$$|L_\delta f(x) - f(x)| \leqslant c_\delta(x)\, \omega_\varphi^2(f; \delta)_\infty, \qquad x \in I, \quad 0 < \delta < \delta_0, \quad f \in \mathscr{M}(\varphi), \tag{3}$$

where $c_\delta(x)$ may depend on δ and x, but not upon f. The inequality

$$\sup_{0<u\leqslant\delta} \|L_u f - f\| \leqslant C\omega_\varphi^2(f; \delta)_\infty, \qquad 0 < \delta < \delta_0, \quad f \in \mathscr{M}(\varphi), \tag{4}$$

where $\|\cdot\|$ stands for the usual sup-norm and C is an absolute constant is called a direct inequality.

Several authors have obtained estimates of the constants $c_\delta(x)$ and C for the ordinary second modulus of continuity, i.e., for $\varphi \equiv 1$. For instance, with regard to the Bernstein polynomials, Gonska [8] showed that $1 \leqslant C \leqslant 3.25$ and Păltănea [11] obtained $C = 1.094$. Pointwise-type estimates for this operator can be found in Gonska and Zhou [9] and Kacsó [10].

Although $\omega_\varphi^2(f; \cdot)_\infty$ gives the right order of uniform convergence, the authors do not know explicit estimates of the constants $c_\delta(x)$ and C in (3)–(4) with respect to a general weight function φ. Under mild assumptions on φ—close to those assumed in [5, 13]—we obtain general pointwise estimates which are applied to some classical families of operators (Section 3). If these assumptions are not fulfilled, we provide in Section 4 a counterexample of a family of operators of the form (1)–(2) not satisfying a direct inequality.

Finally, unless otherwise specified, we fix from now on $\delta > 0$ and write L, $Z(x)$, $Y(x)$ and $c(x)$ instead of L_δ, $Z_\delta(x)$, $Y_\delta(x)$, and $c_\delta(x)$, respectively. Accordingly, we shall consider inequalities of the form

$$|Lf(x) - f(x)| \leqslant c(x)\, \omega_\varphi^2(f; \delta)_\infty, \qquad x \in I, \quad f \in \mathcal{M}(\varphi). \tag{5}$$

2. TECHNICAL LEMMAS

In some classical examples, the weight function φ vanishes at the finite endpoints of I. This implies—in contraposition with the ordinary second modulus—that we cannot compare $\omega_\varphi^2(f; h)_\infty$ for different values of h. To solve this problem, denote by $H_x := \{h \geqslant 0 : B(x, h) \subseteq I\}$, $x \in I^0$. If $A \subseteq I$, we set $m(A) := \inf\{\varphi(x) \colon x \in A\}$. For any $x \in I^0$ and $h \in H_x \backslash \{0\}$, we define

$$K(x, h) := \inf \left\{ k = 1, 2, \ldots : \frac{h}{k\, m\left(B\left(x, \frac{k-1}{k} h\right)\right)} \leqslant \delta \right\} \; (K(x, 0) = 0). \tag{6}$$

Lemma 2.1. *Let $f \in \mathcal{M}(\varphi)$, $x \in I^0$ and $h \in H_x$. Then*

$$|\Delta_h^2 f(x)| \leqslant K^2(x, h)\, \omega_\varphi^2(f; \delta)_\infty.$$

Proof. Let $h \in H_x \backslash \{0\}$ and $k = 1, 2, \ldots$ be fixed. Denote by $u_r = x - h + rh/k$, $r = 0, \ldots, 2k$. Let $a_r = r$, if $r = 1, \ldots, k-1$ and $a_r = 2k - r$, if $r = k, \ldots, 2k-1$. Then

$$|\Delta_h^2 f(x)| = \left| \sum_{r=1}^{2k-1} a_r \Delta_{h/k}^2 f(u_r) \right| \leqslant k^2 \omega_\varphi^2 \left(f; \frac{h}{k\, m\left(B\left(x, \frac{k-1}{k} h\right)\right)} \right)_\infty.$$

Since k is arbitrary, the conclusion follows from (6). ∎

Remark 2.1. Denote by $\lceil x \rceil$ the ceiling of x, i.e., $\lceil x \rceil := \inf\{k \in \mathbb{Z} : k \geqslant x\}$. It follows from (6) that for $\varphi \equiv 1$, $K(x, h) = \lceil h/\delta \rceil$ and, therefore, Lemma 2.1 extends the well-known inequality $\omega_1^2(f; h)_\infty \leqslant \lceil h/\delta \rceil^2\, \omega_1^2(f; \delta)_\infty$.

Remark 2.2. Let $x \in I^0$ and $h \in H_x$. Clearly, (6) implies that $K(x, \cdot)$ is nondecreasing and $K(x, h) = 1$, whenever $0 < h \leqslant \delta\varphi(x)$. Also,

$$K(x, h) \leqslant \left\lceil \frac{h}{\delta m(B(x, h))} \right\rceil \qquad \left(\frac{1}{0} = +\infty\right).$$

Thus, $K(x, h) < \infty$ if $I = \mathbb{R}$ or if $h \in H_x^0$. Otherwise, $K(x, h)$ may not be finite. For instance, $K(x, x) = \infty$ if $I = [0, \infty)$, $\varphi(x) = x$ and $\delta < 1$.

Sufficient conditions ensuring the finiteness of $K(x, \cdot)$ are given in the following Lemma. To this end, let $I = [0, \infty)$ or $I = [0, 1]$. Denote by $d(\cdot, \cdot)$ the usual euclidean distance and by $I^c := \mathbb{R} \backslash I$. For any $x \in I^0$, we set

$$J_x := \{ y \in I^0 : d(y, I^c) \geqslant d(x, I^c) \} \qquad \text{and} \qquad m(x) := m(J_x). \tag{7}$$

We shall assume that

(i) $m(\cdot)$ is positive on I^0 and the function

$$r(x) := \frac{d(x, I^c)}{m(x)}, \qquad x \in I^0 \tag{8}$$

satisfies the following: $r(x) \to 0$, as $x \to 0$, and $r(\cdot)$ is non-decreasing on $(0, \infty)$ or on $(0, 1/2]$, according to $I = [0, \infty)$ or $I = [0, 1]$.

LEMMA 2.2. *Let $I = [0, \infty)$ or $I = [0, 1]$, $x \in I^0$ and $0 < h \leqslant d(x, I^c)$. If condition* (i) *above is satisfied, then*

$$K(x, h) \leqslant 1 \vee \left\lceil \frac{h}{a} \right\rceil < \infty \qquad \left(\frac{1}{+\infty} = 0 \right), \tag{9}$$

where $\vee$ stands for maximum and

$$a := \sup \{ z \in I^0 : 2z \in I^0, r(z) \leqslant \delta \}. \tag{10}$$

Proof. Let $x \in I^0$ and $0 < h \leqslant d(x, I^c)$. By assumption (i), $0 < a \leqslant \infty$. If $I = [0, \infty)$ and $a = \infty$, then $0 < h \leqslant d(x, I^c) \leqslant \delta m(x) \leqslant \delta \varphi(x)$ and, therefore, $K(x, h) = 1$, as follows from Remark 2.2. Otherwise, since $B(x, (k-1) h/k) \subseteq J_{h/k}$, $k = 1, 2, ...,$ we have from (6) and (10)

$$K(x, h) \leqslant \inf \left\{ k = 1, 2, ... : \frac{h/k}{m(h/k)} \leqslant \delta \right\} = \inf \left\{ k = 1, 2, ...: \frac{h}{k} \leqslant a \right\}.$$

This completes the proof of Lemma 2.2. ∎

As pointwise approximants of any function $f \in C(I)$, we consider the second order Steklov means of f given by

$$P_h f(x) := \frac{1}{4h^2} \int_{-h}^{h} \int_{-h}^{h} f\left(x + \frac{u+v}{2} \right) du \, dv, \qquad x \in I^0, \quad h \in H_x \backslash \{0\}.$$

An immediate consequence of Lemma 2.1 is the following.

LEMMA 2.3. *Let $f \in \mathcal{M}(\varphi)$, $x \in I^0$, and $h \in H_x \backslash \{0\}$. We have*

(a) $P_h f(x) = (1/h^2)\, \Delta_h^2 F(x)$, *with* $F''(x) = f(x)$.

(b) $|P_h f(x) - f(x)| \leqslant (1/2)\, K^2(x, h)\, \omega_\varphi^2(f; \delta)_\infty$.

(c) $|(P_h f)''(x)| \leqslant (1/h^2)\, K^2(x, h)\, \omega_\varphi^2(f; \delta)_\infty$.

3. DIRECT INEQUALITIES

Let $J \subseteq I^0$ be a closed interval. If $x \in I \backslash J$, we denote by x^* the endpoint of J such that $d(x, J) = d(x, x^*)$, while if $x \in J$, we set $x^* = x$. We say that J is a symmetrization subinterval of I if $2x^* - x \in J$, for every $x \in I$. We denote by 1_A the indicator of the set A and use the convention $d(\cdot, \varnothing) = +\infty$. The direct inequalities are based on the following

THEOREM 3.1. *Let J be a symmetrization subinterval of I such that $m(J) > 0$. For any $x \in I$ and $0 < h \leqslant d(J, I^c) \wedge \delta m(J)$, where $\wedge$ stands for minimum, we have*

$$|Lf(x) - f(x)| \leqslant (c_1(x) + c_2(x) + c_3(x))\, \omega_\varphi^2(f; \delta)_\infty, \qquad f \in \mathcal{M}(\varphi),$$

where

$$c_1(x) = EK^2(Z^*(x), |Z(x) - Z^*(x)|)\, 1_{\{Z(x) \notin J\}} + K^2(x^*, |x - x^*|),$$

$$c_2(x) = 1 + P(Z(x) \notin J, Z^*(x) \neq x^*)\, 1_{\{x = x^*\}}$$
$$+ (P(Z(x) \in J) + 2P(Z(x) \notin J, Z^*(x) \neq x^*))\, 1_{\{x \neq x^*\}}$$

and

$$c_3(x) = \frac{1}{h^2}\left(\frac{1}{2}\, E(Z(x) - x)^2 + (x - x^*)^2 + E(Z^*(x) - x^*)^2\, 1_{\{Z(x) \notin J\}}\right).$$

Proof. Fix $x \in I$ and call $Z = Z(x)$. Since $h \leqslant d(J, I^c)$, we can write

$$f(Z) - f(x) = I + II + III,$$

where

$$I = f(Z) - 2f(Z^*) + f(2Z^* - Z) - (f(x) - 2f(x^*) + f(2x^* - x)),$$

$$II = ((f - P_h f)(Z^*) - (f - P_h f)(x^*))$$
$$+ ((f - P_h f)(Z^*) - (f - P_h f)(2Z^* - Z))$$
$$- ((f - P_h f)(x^*) - (f - P_h f)(2x^* - x))$$

and

$$III = (P_h f(Z^*) - P_h f(x^*)) + (P_h f(Z^*) - P_h f(2Z^* - Z)) - (P_h f(x^*) - P_h f(2x^* - x)).$$

The constant $c_1(x)$ is obtained taking expectations in I and applying Lemma 2.1 (recall that $K(Z^*, 0) = 0$). On the other hand, since $h \leqslant d(\xi, I^c) \wedge \delta\varphi(\xi)$, for any $\xi \in J$, we have from Remark 2.2

$$K(\xi, h) \leqslant 1, \qquad \xi \in J. \tag{11}$$

Therefore, the constant $c_2(x)$ is obtained taking expectations in II and applying (11) and Lemma 2.3(b).

Finally, expanding III in a Taylor series around x^* and taking into account that $EZ = x$, Lemma 2.3(c) and (11), we have

$$|E(III)| \leqslant \frac{1}{h^2} \omega_\varphi^2(f; \delta)_\infty \left(\frac{1}{2} E(Z - x^*)^2 \, 1_{\{Z \in J\}} + E\left[(Z^* - x^*)^2 + \frac{1}{2} (2Z^* - Z - x^*)^2 \right] 1_{\{Z \notin J\}} + \frac{1}{2} (x - x^*)^2 \right).$$

The constant $c_3(x)$ is obtained from this inequality, after observing that on the event $\{Z \notin J\}$, we have

$$(2Z^* - Z - x^*)^2 = 4(Z^* - x^*)^2 - 4(Z^* - x^*)(Z - x^*) + (Z - x^*)^2 \leqslant (Z - x^*)^2,$$

and noting that $E(Z - x^*)^2 = E(Z - x)^2 + (x - x^*)^2$, since $EZ = x$. The proof of Theorem 3.1 is complete. ▮

Pointwise estimates of the form (5) are given in the following two results.

COROLLARY 3.1. *Let $I = \mathbb{R}$. If $m(I) > 0$, then inequality (5) holds with*

$$c(x) = 1 + \frac{1}{2} E\left(\frac{\varphi(x)\, Y(x)}{m(I)} \right)^2, \qquad x \in I.$$

Proof. It is enough to choose in Theorem 3.1 $J = I$ and $h = \delta m(I)$, so that $Z^*(x) = Z(x)$ and $x^* = x$, for any $x \in I$. ▮

To deal with the cases $I = [0, \infty)$ or $I = [0, 1]$, we shall assume that a, as defined in (10), satisfies the following property:

(ii) $a < \infty$ or $a \leqslant 1/3$, according to $I = [0, \infty)$ or $I = [0, 1]$.

Observe that under assumption (i) preceding Lemma 2.2, condition (ii) is fulfilled for $\delta < \lim_{x \to \infty} r(x)$ or $\delta \leqslant r(\frac{1}{3})$, according to $I = [0, \infty)$ or $I = [0, 1]$.

COROLLARY 3.2. *Let $I = [0, \infty)$ or $I = [0, 1]$ and let J_x be as in* (7). *If conditions* (*i*) *and* (*ii*) *above are satisfied, then inequality* (5) *holds with*

(a) *If $x \notin J_a$, then*

$$c(x) = 3 + \frac{1}{2} E\left(\frac{\varphi(x)\, Y(x)}{m(a)}\right)^2 + \left(\frac{x - x^*}{a}\right)^2 + \left(2 + \frac{(1-2a)^2}{a^2}\right) P(Z(x) \notin J_a, Z^*(x) \neq x^*).$$

(b) *If $x \in J_a$, then for any symmetrization subinterval J_b of I such that $x \in J_b \subseteq J_a$, we have*

$$c(x) = 1 + \frac{1}{2} E\left(\frac{\varphi(x)\, Y(x)}{m(b)}\right)^2 + E\left(1 + \left\lceil \frac{d(b, I^c)}{a} \right\rceil^2 + \left(\frac{Z^*(x) - x}{\delta m(b)}\right)^2\right) 1_{\{Z(x) \notin J_b\}}.$$

Proof. (a) By condition (ii), J_a is a symmetrization subinterval of I. Choose in Theorem 3.1 $J = J_a$ and $h = a = \delta m(a)$, so that $|x - x^*| \leqslant a$ and $|Z(x) - Z^*(x)| \leqslant a$. By Remark 2.2, $c_1(x) = P(Z(x) \notin J_a) + 1$. On the other hand, since $x \notin J_a$, $x^* \neq x$ and, therefore,

$$c_2(x) = 1 + P(Z(x) \in J_a) + 2P(Z(x) \notin J_a, Z^*(x) \neq x^*).$$

Finally, to bound $c_3(x)$, note that $E(Z^*(x) - x^*)^2\, 1_{\{Z(x) \notin J_a\}}$ is positive only if $I = [0, 1]$, on the event $\{Z^*(x) = 1 - x^*\}$. Hence,

$$E(Z^*(x) - x^*)^2\, 1_{\{Z(x) \notin J_a\}} = (1 - 2a)^2\, P(Z(x) \notin J_a, Z^*(x) \neq x^*),$$

thus completing the proof of part (a).

(b) Choose in Theorem 3.1 $J = J_b$ and $h = \delta m(b) \leqslant d(b, I^c)$, as follows from assumption (i). In this case, $x^* = x$. Therefore, by Lemma 2.2

$$c_1(x) \leqslant \left\lceil \frac{d(b, I^c)}{a} \right\rceil^2 P(Z(x) \notin J_b) \qquad \text{and} \qquad c_2(x) \leqslant 1 + P(Z(x) \notin J_b).$$

Since the bound for $c_3(x)$ is trivial, this completes the proof of Corollary 3.2. ∎

To illustrate the preceding results, we consider the following examples.

EXAMPLE 3.1. *The Weierstrass operator.* For any fixed $\sigma > 0$, let $(W_t^\sigma, t > 0)$ be the family of operators defined as

$$W_t^\sigma f(x) := Ef\left(x + \sigma \frac{W(t)}{t}\right) = \frac{\sqrt{t}}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x+\theta)\, e^{-t\theta^2/2\sigma^2}\, d\theta, \qquad x \in \mathbb{R},$$

where $(W(t), t \geqslant 0)$ is the standard Brownian motion. Observe that W_t^1 is the classical Weierstrass operator. Since $\delta = 1/\sqrt{t}$ and $\varphi(x) \equiv 1$, we immediately obtain from Corollary 3.1

COROLLARY 3.3. *Let $f \in \mathcal{M}(\varphi)$, with $\varphi(x) = 1$, $x \in \mathbb{R}$. Then,*

$$\| W_t^\sigma f - f \| \leqslant \left(1 + \frac{\sigma^2}{2}\right) \omega_\varphi^2 \left(f; \frac{1}{\sqrt{t}}\right)_\infty, \qquad t > 0.$$

The constant in Corollary 3.3 can also be obtained by applying [11, Theorem 2.1]. On the other hand, using the symmetry of each random variable $W(t)$ and Remark 2.1, we obtain the estimate

$$\frac{1}{2} E\left(1 + \sigma \frac{|W(t)|}{\sqrt{t}}\right)^2 = \frac{1}{2} + \sigma \sqrt{\frac{2}{\pi}} + \frac{\sigma^2}{2}.$$

This was pointed out in [3, p. 137] for the Gauss–Weierstrass integral ($\sigma = \sqrt{2}$). Note that Corollary 3.3 provides better constants if $\sigma \geqslant \sqrt{\pi}/(2\sqrt{2})$.

EXAMPLE 3.2. *The Szàsz operator.* Let S_t be the operator defined as

$$S_t f(x) := Ef\left(\frac{N(tx)}{t}\right) = \sum_{k=0}^{\infty} f\left(\frac{k}{t}\right) e^{-tx} \frac{(tx)^k}{k!}, \qquad x \geqslant 0, \quad t > 0,$$

where $(N(y), y \geqslant 0)$ is the standard Poisson process. We shall need the following exponential bound, the proof of which follows along the lines of that in [12, p. 52]

$$P(N(y) \leqslant \beta y) \leqslant \exp(-g(\beta)\, y), \qquad y \geqslant 0, \quad 0 \leqslant \beta \leqslant 1, \tag{12}$$

where $g(\cdot)$ is the convex function given by

$$g(\beta) := 1 - \beta + \beta \log \beta, \qquad 0 \leqslant \beta \leqslant 1. \tag{13}$$

COROLLARY 3.4. *Let $f \in \mathscr{M}(\varphi)$, with $\varphi(x)=\sqrt{x}$, $x \geqslant 0$. Then,*

$$\|S_t f - f\| \leqslant 4\omega_\varphi^2\left(f; \frac{1}{\sqrt{t}}\right)_\infty, \qquad t>0.$$

Proof. First, we can write for any $x>0$ and $t>0$

$$\frac{N(tx)}{t} = x + \sqrt{\frac{x}{t}}\, Y_t(x), \qquad Y_t(x) := \frac{N(tx)-tx}{\sqrt{tx}}, \qquad \text{and}$$

$$EY_t^2(x)=1. \tag{14}$$

We therefore have $\delta = 1/\sqrt{t}$, $\varphi(x)=m(x)=\sqrt{x}$ and $a=1/t$. Fix $x>0$ and $t>0$ and denote by $y := tx$. We distinguish the following cases:

Case $y<1$. Corollary 3.2(a) and (14) yield

$$c(y) = 3 + \tfrac{1}{2}y + (y-1)^2 \leqslant 4.$$

Case $y \geqslant 1$. We apply (14) and Corollary 3.2 (b) by choosing $b=\beta x$, with $1/y \leqslant \beta \leqslant 1$, to obtain

$$c(y) = 1 + \frac{1}{2\beta} + \left(1 + \ulcorner \beta y \urcorner^2 + \frac{(1-\beta)^2}{\beta}\, y\right) P(N(y) < \beta y). \tag{15}$$

According to $1 \leqslant y < 5$ or $5 \leqslant y < 10$, we choose $\beta = i/y$, $i=1, 2$ in (15), thus obtaining the bound

$$c_i(y) := 1 + \frac{y}{2i} + \left(1 + i^2 + \frac{(y-i)^2}{i}\right) e^{-y} \sum_{k=0}^{i-1} y^k \leqslant c(5^-) = 3.62128,$$

as follows from the convexity of each $c_i(\cdot)$ on its respective domain of definition. Finally, for $y \geqslant 10$, we choose $\beta = 1/4$ in (15) and apply (12) to obtain the bound

$$c(y) \leqslant 3 + \left(1 + \left(\frac{y}{4}+1\right)^2 + \frac{9y}{4}\right) e^{-g(1/4)\,y} \leqslant c(10) = 3.63273,$$

the last inequality because the function $h(y) := y^p e^{-\alpha y}$ $(p>0,\ \alpha>0)$ is decreasing for $y \geqslant p/\alpha$. The proof of Corollary 3.4 is complete. ∎

EXAMPLE 3.3. *Bernstein polynomials.* For $n=1, 2, ...,$ let B_n be the operator defined as

$$B_n f(x) := Ef\left(\frac{S_n(x)}{n}\right) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}, \qquad 0 \leqslant x \leqslant 1,$$

where

$$S_n(x)=\sum_{k=0}^{n} 1_{\{0\leqslant U_k\leqslant x\}}, \qquad 0\leqslant x\leqslant 1, \quad n=1, 2, ...$$

$(U_k, k=1, 2, ...)$ being a sequence of independent random variables uniformly distributed on $[0, 1]$. As in (12), we have for $n=1, 2, ...,$

$$P(S_n(x)\leqslant \beta nx)\leqslant \exp(-g(\beta)\, nx), \qquad 0\leqslant x\leqslant 1, \quad 0\leqslant \beta\leqslant 1, \tag{16}$$

where $g(\beta)$ is defined in (13).

COROLLARY 3.5. *Let* $f\in \mathscr{M}(\varphi)$, *with* $\varphi(x)=\sqrt{x(1-x)}$, $x\in[0, 1]$. *Then,*

$$\|B_n f-f\|\leqslant 4\omega_\varphi^2\left(f; \frac{1}{\sqrt{n}}\right)_\infty, \qquad n=1, 2, ...$$

Proof. Since the random variables $S_n(1-x)$ and $n-S_n(x)$, have the same law, we can restrict our attention to $x\in(0, 1/2]$. On the other hand, we can write for $0<x<1$ and $n=1, 2, ...$

$$\frac{S_n(x)}{n}=x+\sqrt{\frac{x(1-x)}{n}}\, Y_n(x), \qquad Y_n(x):=\frac{S_n(x)-nx}{\sqrt{nx(1-x)}},$$
$$EY_n^2(x)=1. \tag{17}$$

Thus, we have $\delta=1/\sqrt{n}$, $\varphi(x)=m(x)=\sqrt{x(1-x)}$, and $a=1/(n+1)$. As $B_1 f$ is linear and interpolates f at 0 and 1, we have from [2] that

$$\|B_1 f-f\|\leqslant \omega_1^2(f; \tfrac{1}{2})_\infty\leqslant \omega_\varphi^2(f; 1)_\infty.$$

Denote by $y:=nx$. We distinguish the following cases:

Case $0<y<n/(n+1)$. From Corollary 3.2(a) and (17), we obtain

$$c_n(y)=3+\frac{(n+1)^2\, y(n-y)}{2n^3}+\left(\frac{(n+1)\, y}{n}-1\right)^2$$
$$+(2+(n-1)^2)\left(\frac{y}{n}\right)^n\leqslant 4,$$

since $c_n(\cdot)$ is convex and attains its maximum at $y=0$.

Case $n/(n+1) \leqslant y \leqslant n/2$. We apply (17) and Corollary 3.2(b) by choosing $b = \beta x$, with $n/((n+1)\, y) \leqslant \beta \leqslant 1$, to obtain

$$\begin{aligned} c_n(y) &= 1 + \frac{n-y}{2\beta(n-\beta y)} + \left(1 + \left\lceil \frac{(n+1)\,\beta y}{n} \right\rceil^2 + \frac{(1-\beta)^2\, ny}{\beta(n-\beta y)}\right) \\ &\quad \times P\left(S_n\left(\frac{y}{n}\right) < \beta y\right) + \left(1 + \left\lceil \frac{(n+1)\,\beta y}{n} \right\rceil^2 + \frac{(n-y-\beta y)^2\, n}{(n-\beta y)\,\beta y}\right) \\ &\quad \times P\left(S_n\left(1-\frac{y}{n}\right) < \beta y\right). \end{aligned} \tag{18}$$

If $2 \leqslant n \leqslant 12$ or if $n/(n+1) \leqslant y < 1$, it can be checked that $c_n(y) < 4$, by choosing $\beta = n/((n+1)\, y)$ in (18). Therefore, we can assume in what follows that $n \geqslant 13$.

If $1 \leqslant y < 5$, we set $\beta = 1/y$ in (18). Since $S_n(1 - \frac{y}{n}) \leqslant S_n(\frac{1}{2})$, we have the bound

$$\begin{aligned} c_n(y) \leqslant A_n(y) &:= 1 + \frac{y}{2} + \left(5 + \frac{n}{n-1}\,(y-1)^2\right) e^{-y} \\ &\quad + \left(5 + \frac{n(n-2)^2}{n-1}\right) \frac{1}{2^n} \leqslant A_{13}(5) = 3.66709, \end{aligned}$$

where we have used the monotonicity of $A_{\cdot}\,(y)$ and the convexity of $A_{13}(\cdot)$.

If $5 \leqslant y < 10 \wedge n/2$, we set $\beta = 2/y$ in (18), thus obtaining

$$\begin{aligned} c_n(y) \leqslant B_n(y) &:= 1 + \frac{y}{4} + \left(10 + \frac{n}{2(n-2)}\,(y-2)^2\right)(1+2y)\, e^{-y} \\ &\quad + \left(10 + \frac{n(n-7)^2}{2(n-2)}\right) \frac{n+1}{2^n} \leqslant B_{13}(10) = 3.59903, \end{aligned}$$

as follows from the convexity of $B_{13}(\cdot)$.

Finally, if $10 \leqslant y \leqslant n/2$, we set $\beta = 1/4$ in (18) and use the exponential bound (16) to obtain

$$\begin{aligned} c_n(y) \leqslant C_n(y) &:= 1 + 8\,\frac{n-y}{4n-y} + \left(1 + \left(\frac{(n+1)\,y}{4n} + 1\right)^2 + \frac{18y}{7}\right) e^{-g(1/4)\,y} \\ &\quad + \left(1 + \left(\frac{n+1}{8} + 1\right)^2 + \frac{(4n-5y)^2\, n}{(4n-y)\,y}\right) e^{-g(1/4)\,n/2} \leqslant C_n(10) \leqslant 4, \end{aligned}$$

where we have used the monotonicity of $C_n(\cdot)$ for $n \geqslant 13$. This completes the proof of Corollary 3.5. ∎

4. CONCLUDING REMARKS

As shown in Lemma 2.2, assumption (i) guarantees the finiteness of $K(x, \cdot)$, which is a basic ingredient in obtaining direct inequalities. Without this assumption, $K(x, \cdot)$ may not be finite (recall Remark 2.2), but in such a case, a direct inequality is not always valid, not even a pointwise estimate as that in (5). To see this, we give the following

EXAMPLE 4.1. Let $I = [0, \infty)$ and let φ be a non-decreasing weight function, so that $\varphi(\cdot) \equiv m(\cdot)$. Suppose that $r(\cdot)$, as defined in (8), is non-decreasing, and that $r(x) \to r > 0$, as $x \to 0$.

For any $t > r^{-2}$, we consider the operator L_t on $[0, \infty)$ defined as

$$L_t f(x) := Ef(Z_t(x)), \qquad x \geqslant 0,$$

where the random variable $Z_t(x)$ takes on the values 0, x and $2x$, with probabilities

$$p_t(x) := P(Z_t(x) = 0) = P(Z_t(x) = 2x)$$
$$= \frac{1}{2}(1 - P(Z_t(x) = x)) = \frac{1}{2tr^2(x)}.$$

The preceding family of operators $\mathbb{L} := (L_t, t > r^{-2})$ has the form given by (1)–(2), with $\delta = 1/\sqrt{t}$, $\varphi(x)$ as above and $EY_E^2 t(x) = 1$. Fix $t > r^{-2}$ and $x > 0$. For any $\varepsilon > 0$, the function $f_\varepsilon(y) := \log(y \vee \varepsilon)$ satisfies

$$\omega_\varphi^2\left(f_\varepsilon; \frac{1}{\sqrt{t}}\right)_\infty \leqslant \log \frac{r\sqrt{t}+1}{r\sqrt{t}-1}, \tag{19}$$

since the first order increments of $f_\varepsilon(y)$ are bounded by the first order increments of $\log y$. On the other hand,

$$|L_t f_\varepsilon(x) - f_\varepsilon(x)| = |\log \varepsilon + \log 2 - \log x| \; p_t(x), \qquad 0 < \varepsilon < x. \tag{20}$$

ε being arbitrary, (19) and (20) prove that inequality (5) does not hold.

REFERENCES

1. J. A. Adell and J. de la Cal, Bernstein-type operators diminish the ϕ-variation, *Constr. Approx.* **12** (1996), 489–507.
2. H. Burkill, Cèsaro–Perron almost periodic functions, *Proc. London Math. Soc. (3)* **2** (1952), 150–174.
3. P. L. Butzer and R. L. Nessel, "Fourier Analysis and Approximation," Vol. I, Birkhäuser, Basel, 1971.

4. W. Chen and Z. Ditzian, Strong converse inequality for Kantorovich polynomials, *Constr. Approx.* **10** (1994), 107–129.
5. Z. Ditzian, Direct estimates for Bernstein polynomials, *J. Approx. Theory* **79** (1994), 165–166.
6. Z. Ditzian and V. Totik, “Moduli of Smoothness,” Springer-Verlag, New York, 1987.
7. Z. Ditzian and K. G. Ivanov, Strong converse inequalities, *J. Anal. Math.* **61** (1993), 61–111.
8. H. H. Gonska, Two problems on best constants in direct estimates, *in* “Problem Section of Proc. Edmonton Conf. Approximation Theory” (Z. Ditzian *et al.*, Eds.), pp. 194, Amer. Math. Soc., Providence, 1983.
9. H. H. Gonska and D. X. Zhou, On an extremal problem concerning Bernstein operators, *Serdica Math. J.* **21** (1994), 137–150.
10. D. P. Kacsó, Approximation by means of piecewise linear functions, *Results Math.* **35** (1999), 89–102.
11. R. Păltănea, Best constants in estimates with second order moduli of continuity, *in* “Approximation Theory, Proc. Int. Dortmund Meeting on Approximation Theory, 1995” (M. W. Müller *et al.*, Eds.), pp. 251–275, Akademie Verlag, Berlin, 1995.
12. V. V. Petrov, “Sums of Independent Random Variables,” Springer-Verlag, Berlin, 1975.
13. V. Totik, Strong converse inequalities, *J. Approx. Theory* **76** (1994), 369–375.